AD-A178 742 A MULTIVARIATE TEST TO CHECK THE VALIDITY OF A CERTAIN STRUCTURE OF THE C..(U) FLORIDA UNIV GAINESVILLE DEPT OF STATISTICS A I KHURI MAR 87 TR-280 N00014-86-K-0059 1/1

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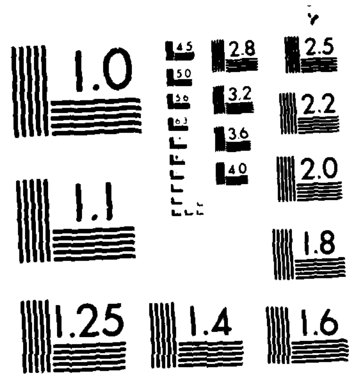

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A MULTIVARIATE TEST TO CHECK THE VALIDITY OF A CERTAIN STRUCTURE OF THE COVARIANCE MATRIX IN SCHEFFE'S MIXED MODEL

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Technical Report Number 280

March 1987

PREPARED UNDER GRANT NO.N00014-86-K-0059 FROM THE OFFICE OF NAVAL RESEARCH. ANDRE' I. KHURI AND RAMON C. LITTELL, PRINCIPAL INVESTIGATORS.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 280	2. GOVT ACCESSION NO. AD-A178742	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Multivariate Test to Check the Validity of a Certain Structure of the Covariance Matrix in Scheffé's Mixed Model		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. I. Khuri		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0059 (R & T 4114552---01) Acct. No. 4910-1623459
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Nuclear Sciences Center University of Florida; Gainesville, FL 32611		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Mathematical Sciences Division (Code 411) Arlington, VA 22217-5000		12. REPORT DATE March 1987
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release: Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Unbalanced data; Mixed two-way model; Random effects.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a multivariate test to check the validity of the usual assumptions made concerning the random effects in an unbalanced mixed two-way classification model.

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S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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A. I. KHURI*

ABSTRACT

In this article, Scheffe´'s mixed two-way classification model for balanced as well as unbalanced data is considered. The covariance structure associated with this model's random effects is quite general. Under the usual assumptions of independence of the random effects and homogeneity of their respective variances, the covariance structure assumes a particular form. A multivariate test is presented to check the validity of this form. A numerical example is given to illustrate the implementation of the proposed test.

KEY WORDS: Balanced data; Unbalanced data; Mixed two-way classification model; Model I; Model II; Random effects.

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1. INTRODUCTION

Different sets of assumptions can be found in the statistical literature concerning the random effects in a mixed linear model. A discussion of these assumptions with regard to the balanced two-way classification model was given by Hocking (1973). the least restrictive of these sets of assumptions is the one considered by Scheffe´ (1956; 1959, Chapter 8). In another set of assumptions that is commonly found in the literature, the random effects are described as being statistically independent and normally distributed with zero means and variances that are equal for all levels associated with a particular random effect. We shall refer to the mixed two-way classification model as Model II or as Model I depending on whether the latter assumptions or those considered by Scheffe´ are valid, respectively.

Let us consider the unbalanced mixed two-way classification model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \qquad (1.1)$$
$$i=1,2,\ldots,r; \quad j=1,2,\ldots,s; \quad k=1,2,\ldots,n_{ij}$$

where μ and α_i are fixed unknown parameters; β_j, $(\alpha\beta)_{ij}$, and ϵ_{ijk} are random variables assumed to be normally distributed with zero means. For reasons to be seen later, it is also assumed that $n_{..} > 2rs$, where $n_{..} = \Sigma_{i,j}n_{ij}$ is the total number of observations.

Let $\bar{y}_{ij.}$ be the $(i,j)^{th}$ cell sample mean $(i=1,2,\ldots,r;$ $j=1,2,\ldots,s)$. From (1.1) we have

$$\bar{y}_{ij.} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \bar{\varepsilon}_{ij.}, \quad \begin{array}{l} i=1,2,\ldots,r; \\ j=1,2,\ldots,s, \end{array} \tag{1.2}$$

where

$$\bar{\varepsilon}_{ij.} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} \varepsilon_{ijk}.$$

Model (1.2) can be written in vector form as

$$\bar{\underset{\sim}{y}}_j = \mu \underset{\sim}{1}_r + \underset{\sim}{\alpha} + \underset{\sim}{\theta}_j + \bar{\underset{\sim}{\varepsilon}}_j, \quad j=1,2,\ldots,s, \tag{1.3}$$

where

$\bar{\underset{\sim}{y}}_j = (\bar{y}_{1j.}, \bar{y}_{2j.}, \ldots, \bar{y}_{rj.})'$, $\quad j=1,2,\ldots,s,$

$\underset{\sim}{1}_r$ is a column vector of ones of order $r \times 1$,

$\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)'$,

$\underset{\sim}{\theta}_j = \beta_j \underset{\sim}{1}_r + (\underset{\sim}{\alpha\beta})_j$,

where

$(\underset{\sim}{\alpha\beta})_j = [(\alpha\beta)_{1j}, (\alpha\beta)_{2j}, \ldots, (\alpha\beta)_{rj}]'$, $\quad j=1,2,\ldots,s,$

$\bar{\underset{\sim}{\varepsilon}}_j = (\bar{\varepsilon}_{1j.}, \bar{\varepsilon}_{2j.}, \ldots, \bar{\varepsilon}_{rj.})'$, $\quad j=1,2,\ldots,s.$

Under Model I, the $\underset{\sim}{\theta}_j$ and $\bar{\underset{\sim}{\varepsilon}}_j (j=1,2,\ldots,s)$ are independently distributed as $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ and $N(\underset{\sim}{0}, \sigma_\varepsilon^2 \underset{\sim}{K}_j)$, respectively, whre $\underset{\sim}{\Sigma}$ is the variance-covariance matrix of $\underset{\sim}{\theta}_j$, σ_ε^2 is the error variance, and $\underset{\sim}{K}_j$

is the diagonal matrix

$$\underset{\sim}{K}_j = \mathrm{diag}(n_{1j}^{-1}, n_{2j}^{-1}, \ldots, n_{rj}^{-1}), \quad j=1,2,\ldots,s. \tag{1.4}$$

Under Model II, which is a special case of Model I, $\underset{\sim}{\Sigma}$ has the form

$$\underset{\sim}{\Sigma} = \sigma_\beta^2 \underset{\sim}{J}_r + \sigma_{\alpha\beta}^2 \underset{\sim}{I}_r, \tag{1.5}$$

where σ_β^2 and $\sigma_{\alpha\beta}^2$ are the variances of the β_j and the $(\alpha\beta)_{ij}$, respectively, and $\underset{\sim}{J}_r$ and $\underset{\sim}{I}_r$ are, respectively, the matrix of ones and the identity matrix, both of order $r \times r$.

Quite often, when Model II's assumptions are stated in the literature, no attempt is made to check their validity for the data under cosideration. As was mentioned earlier, the difference between Model I and Model II is in the structure of the variance-covariance matrix, $\underset{\sim}{\Sigma}$. If this structure is of the form given in (1.5), then Model II is considered valid provided, of course, that the other assumptions of Model I are also valid. It is important here to note that Model II improses severe restrictions on the variances and covariances of the random effects. These restrictions may not be easily justified in practice (see Scheffé 1959, pp. 264-265; Alalouf 1980, p. 195). The main attractive feature of these restrictions, particularly in the balanced case, lies in their legitimization of the use of the usual F-ratios for

the traditional analysis of data. Furthermore, these restrictions are superfluous since in the balanced case at least, other exact tests exist for the more flexible Model I as was demonstrated by Scheffé (1959, Chapter 8). The latter tests are not, in general, based on using the F-ratios in the corresponding analysis of variance (ANOVA) table. For example, the test concerning the fixed effects was developed on the basis of Hotelling T^2-statistic. The tests concerning the random effects, however, were obtained in the usual manner using the ANOVA F-ratios. But, unlike Model II, the power of the test for the interaction is not expressible in terms of the central or noncentral F-distribution.

The purpose of this article is to present a test to check the validity of the covariance structure in (1.5) for the unbalanced model in (1.1). More specifically, under the assumptions of Scheffé's Model I, the following hypothesis will be tested:

$$H_0 : \underset{\sim}{\Sigma} = a\underset{\sim}{J}_r + b\underset{\sim}{I}_r, \tag{1.6}$$

where a and b are nonnegative constants. In particular, in case of equal subclass frequencies (that is, when the n_{ij} are equal), the test provides a check on Model II for a balanced data set. The proposed test, which is multivariate in nature, represents a further demonstration of the utility of the multivariate approach to mixed linear models. This approach was discussed in the balanced case by Alalouf (1980).

2. TESTING THE VALIDITY OF THE COVARIANCE STRUCTURE IN (1.6)

Let us again consider the model given in (1.3). Model I will be assumed, that is, the $\underset{\sim}{\theta}_j$ and $\bar{\underset{\sim}{\varepsilon}}_j$ are independently distributed as $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ and $N(\underset{\sim}{0}, \sigma^2_\varepsilon \underset{\sim}{K}_j)$, respectively, where $\underset{\sim}{K}_j$ is the diagonal matrix defined in (1.4). It is also assumed that

$$n_{..} > 2rs, \tag{2.1}$$

where $n_{..}$ is the toal number of observations, r and s are the numbers of levels associated with the main effects.

The mean vector and variance-covariance matrix of $\bar{\underset{\sim}{y}}_j$ in (1.3) are

$$E(\bar{\underset{\sim}{y}}_j) = \mu \underset{\sim}{1}_r + \underset{\sim}{\alpha}, \quad j=1,2,\ldots,s, \tag{2.2}$$

$$\mathrm{Var}(\bar{\underset{\sim}{y}}_j) = \underset{\sim}{\Sigma} + \sigma^2_\varepsilon \underset{\sim}{K}_j, \quad j=1,2,\ldots,s. \tag{2.3}$$

The error sum of squares for model (1.1) is given by

$$Q = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2. \tag{2.4}$$

Formula (2.4) can be rewritten as

$$Q = \underset{\sim}{y}' \underset{\sim}{R} \underset{\sim}{y}, \tag{2.5}$$

where $\underset{\sim}{y} = (\underset{\sim}{y}'_1, \underset{\sim}{y}'_2, \ldots, \underset{\sim}{y}'_s)'$ is the vector of all observations with $\underset{\sim}{y}_j$ being the vector of observations obtained under $j(=1,2,\ldots,s)$. The matrix $\underset{\sim}{R}$ is (2.5) is idempotent of order $n_{..} \times n_{..}$ and rank $n_{..} - rs$, and has the form

$$\underset{\sim}{R} = \underset{\sim}{I}_{n_{..}} - \bigoplus_{i,j} (\underset{\sim}{J}_{n_{ij}} / n_{ij}), \tag{2.6}$$

where $\underset{\sim}{J}_{n_{ij}}$ is the matrix of ones of order $n_{ij}\times n_{ij}$ ($i=1,2,\ldots,r$; $j=1,2,\ldots,s$). The second term on the right-hand side of (2.6) is the direct sum of the $\underset{\sim}{J}_{n_{ij}}/n_{ij}$. Since $\underset{\sim}{R}$ is symmetric with eigenvalues equal to one or zero, it can be expressed as

$$\underset{\sim}{R} = \underset{\sim}{C}\ \mathrm{diag}(\underset{\sim}{I}_{n..-rs}, \underset{\sim}{0})\ \underset{\sim}{C}', \tag{2.7}$$

where "diag" denotes a block-diagonal matrix, $\underset{\sim}{0}$ is a zero matrix of order $rs\times rs$, and $\underset{\sim}{C}$ is an orthogonal matrix of order $n..\times n..$. If condition (2.1) is satisfied, then (2.7) can be rewritten as

$$\underset{\sim}{R} = \underset{\sim}{C}\ \mathrm{diag}(\underset{\sim}{I}_{rs}, \underset{\sim}{I}_{n..-2rs}, \underset{\sim}{0})\underset{\sim}{C}'. \tag{2.8}$$

The matrix $\underset{\sim}{C}$ in (2.8) can be partitioned as

$$\underset{\sim}{C} = [\underset{\sim}{C}_1 : \underset{\sim}{C}_2 : \ldots : \underset{\sim}{C}_s : \underset{\sim}{C}_{s+1} : \underset{\sim}{C}_{s+2}], \tag{2.9}$$

where $\underset{\sim}{C}_i$ is a matrix of order $n..\times r$ for $i=1,2,\ldots,s$; $n..\times(n..-2rs)$ for $i=s+1$; and $n..\times rs$ for $i=s+2$.

Let us now define the vector $\underset{\sim}{w}_j$ as

$$\underset{\sim}{w}_j = \bar{\underset{\sim}{y}}_j + (\lambda\underset{\sim}{I}_r - \underset{\sim}{K}_j)^{\frac{1}{2}}\underset{\sim}{C}_j'\underset{\sim}{y}, \quad j=1,2,\ldots,s, \tag{2.10}$$

where $\bar{\tilde{y}}_j$ and $\tilde{K}_j$ are given in (1.3) and (1.4), respectively, λ is given by

$$\lambda = 1/\left(\min_{i,j}(n_{ij})\right), \tag{2.11}$$

and $\left(\lambda\tilde{I}_r - \tilde{K}_j\right)^{\frac{1}{2}}$ is a diagonal matrix whose i^{th} diagonal element is $(\lambda - 1/n_{ij})^{\frac{1}{2}}$, $i=1,2,\ldots,r$; $j=1,2,\ldots,s$.

The following theorem gives distributional properties of the $\tilde{w}_j$ in (2.10):

Theorem 2.1. Under Model I, the $\tilde{w}_j$ defined in (2.10) have the following properties:

(i) They are statistically indpendent and distributed as normal random vectors.

(ii) $E(\tilde{w}_j) = \tilde{\eta}$, $j=1,2,\ldots,s$.

(iii) $Var(\tilde{w}_j) = \tilde{\Gamma}$, $j=1,2,\ldots,s$,

where

$$\tilde{\eta} = \mu\tilde{1}_r + \tilde{\alpha}, \tag{2.12}$$

$$\tilde{\Gamma} = \tilde{\Sigma} + \lambda\sigma^2_\varepsilon\tilde{I}_r. \tag{2.13}$$

Proof. The proof is given in the Appendix.

It is easy to verify that $\tilde{\Sigma}$ has the form described in (1.6) if and only if $\tilde{\Gamma} = a\tilde{J}_r + \left(b+\lambda\sigma^2_\varepsilon\right)\tilde{I}_r$, which is of the form

$$\underset{\sim}{\Gamma} = a' \underset{\sim}{J}_r + b' \underset{\sim}{I}_r, \tag{2.14}$$

where a′ and b′ are nonnegative constant coefficients. Formula (2.14) can be rewritten in the more familiar form

$$\underset{\sim}{\Gamma} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \cdot & \cdot & \cdot & \rho \\ \rho & 1 & \cdot & \cdot & \cdot & \cdot & \rho \\ \rho & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \rho \\ \rho & \rho & \cdot & \cdot & \cdot & \rho & 1 \end{bmatrix} \tag{2.15}$$

where $\sigma^2 = a' + b'$ and $\rho = a'/(a' + b')$. We note that (2.15) is a special case of the more general expression

$$\underset{\sim}{\Gamma} = \sigma^2 \begin{bmatrix} 1 & \gamma & \gamma & \cdot & \cdot & \cdot & \gamma \\ \gamma & 1 & \cdot & \cdot & \cdot & \cdot & \gamma \\ \gamma & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \gamma \\ \gamma & \gamma & \cdot & \cdot & \cdot & \gamma & 1 \end{bmatrix}, \tag{2.16}$$

where the correlation coefficient, γ, is not necessarily nonnegative as ρ is in (2.15).

According to Theorem 2.1, the r×1 vectors, $\underset{\sim}{w}_1, \underset{\sim}{w}_2, \ldots, \underset{\sim}{w}_s$ form a random smaple of size s from $N(\underset{\sim}{\eta}, \underset{\sim}{\Gamma})$. This sample will be utilized to test the hypothesis

$$H_0' : \underset{\sim}{\Gamma} = \sigma^2 \begin{bmatrix} 1 & \gamma & \gamma & \cdot & \cdot & \cdot & \gamma \\ \gamma & 1 & \cdot & \cdot & \cdot & \cdot & \gamma \\ \gamma & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \gamma \\ \gamma & \gamma & \cdot & \cdot & \cdot & \gamma & 1 \end{bmatrix} . \qquad (2.17)$$

The test is based on the likelihood ratio principle and the test statistic, given by Wilks (1946, page 260) and denoted by L, is such that

$$M = L^{2/s} = \frac{r^r(r-1)^{r-1}|\hat{\underset{\sim}{\Gamma}}|}{(\underset{\sim}{1}_r' \hat{\underset{\sim}{\Gamma}} \underset{\sim}{1}_r)[r \operatorname{tr}(\hat{\underset{\sim}{\Gamma}}) - \underset{\sim}{1}_r'\hat{\underset{\sim}{\Gamma}}\underset{\sim}{1}_r]^{r-1}} . \qquad (2.18)$$

(see also Seber 1984, p. 95). The $r \times r$ matrix, $\hat{\underset{\sim}{\Gamma}}$, in (2.18) is an unbiased estimator of $\underset{\sim}{\Gamma}$ given by

$$\hat{\underset{\sim}{\Gamma}} = \frac{1}{s-1} \sum_{j=1}^{s} (\underset{\sim}{w}_j - \bar{\underset{\sim}{w}})(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}})', \qquad (2.19)$$

where

$$\bar{\underset{\sim}{w}} = \frac{1}{s} \sum_{j=1}^{s} \underset{\sim}{w}_j . \qquad (2.20)$$

In (2.18), $|\cdot|$ and $\operatorname{tr}(\cdot)$ denote the determinant and the trace of a matrix. The null hypothesis in (2.17) is rejected at the α-level of significance if

$$M < \tau_\alpha(r,s), \qquad (2.21)$$

where $\tau_\alpha(r,s)$ is the lower 100α% point of the null distribution of the statistic M in (2.18). The exact null distribution of M was determined by Wilks (1946) for r=2 and r=3. He showed that the probability integral of M for r=2 and that of $\sqrt{M}$ for r=3 are incomplete beta functions with parameters p=(s-2)/2, q=1/2 and p=s-3, q=2, respectively. On that basis, he obtained the percentage points of M for r=2 and r=3 using the beta distribution tables (see Wilks 1946, Table 1, p. 263). Nagarsenker (1975) derived the exact distribution of M and provided a tabulation of the corresponding 5% and 1% points for r=4(1)10 and various values of s. For large values of s and under the null hypothesis (2.17), the statistic,

$$V = -\{s-1-[r(r+1)^2(2r-3)]/[6(r-1)(r^2+r-4)]\}\log(M), \quad (2.22)$$

is asymptotically distributed as chi-square with $[r(r+1)-4]/2$ degrees of freedom (see Box 1950, p. 375; Seber 1984, p. 95). In (2.22), log(M) denotes the natural logarithm of M.

In particular, if the data set is balanced, that is, when $n_{ij} = n$ for all i,j, then from (1.4) and (2.11) we have

$$\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j = 0, \quad j=1,2,\ldots,s.$$

Hence, $\underset{\sim}{w}_j = \bar{\underset{\sim}{y}}_j$ for j=1,2,...,s in (2.10). In this case, the test statistic in (2.18) can be used with $\bar{\underset{\sim}{y}}_j$ substituted for $\underset{\sim}{w}_j$(j=1,2,...,s).

Rejection of the null hypothesis in (2.17) gives support to the contention that Σ is not of the form described in (2.16), hence, it cannot have the form given in (2.15). This in turn implies that Σ is not expressible as in (1.6). Consequently, rejection of the null hypothesis, H_0', in (2.17) is an indication that the null hypothesis, H_0, in (1.6) is false, which can be construed as evidence that Model II is invalid. Failing to reject the null hypothesis in (2.17), however, does not necessarily mean that Model II is valid. This is because this hypothesis does not imply the one in (1.6). In the latter hypothesis, both a and b must be nonnegative. From (2.17) and (2.13) we can only conclude (1.6), but without the nonnegativity constraint on both a and b.

3. A NUMERICAL EXAMPLE

We consider an example given by Hald (1952, p. 471) and reproduced in Scheffé (1959, p. 289, problem 8.2). In this example, measurements were given of the waterproof quality of sheets of material manufactured by $r=3$ different machines (factor A) over a period of $s=9$ days (factor B). Three sheets were selected for each machine-day combination. The response values, namely, the logarithms of the permeability in seconds for the sheets, are given in Table 1. We refer to this balanced data set as Data Set 1. The effects associated with factors A and B are considered fixed and random, respectively.

An unbalanced data set (Data Set 2) was obtained by deleting certain response values from Data Set 1. The deleted values are marked with an asterisk in Table 1. The matrix $\hat{\underset{\sim}{\Gamma}}$ in (2.19) for Data Set 2 has the value

$$\hat{\underset{\sim}{\Gamma}} = \begin{bmatrix} .0239 & .0038 & .0072 \\ .0038 & .0050 & -.0038 \\ .0072 & -.0038 & .0160 \end{bmatrix} .$$

From (2.18) we find that $M = L^{2/9} = .2806$. From Beyer's (1968) tables of percentage points of the beta distribution (Table III.10, p. 256), the corresponding lower 10% point of the statistic M is equal to $(.54744)^2 = .2997$ (recall that for r=3, $\sqrt{M}$ has the beta distribution with parameters $p = s-3 = 6$ and $q = 2$). The test is, therefore, significant at the 10% level. We thus conclude that Model II is invalid for Data Set 2.

Let us consider next Data Set 1 (this is the original balanced data set in Table 1). In this case,

$$\hat{\underset{\sim}{\Gamma}} = \begin{bmatrix} .0288 & .0034 & .0012 \\ .0034 & .0067 & .0005 \\ .0012 & .0005 & .0236 \end{bmatrix} ,$$

and $M = .5737$, which exceeds the 10% critical value, .2997. Consequently, we cannot conclude invalidity of Model II for Data

Set 1 (this does not mean that Model II is valid).

We now show that a slight modification in Data Set 1 can lead to a highly significant test. we shall keep the same subclass frequencies as in Data Set 1, but alter only seven response values in Table 1. The new response values are given in parentheses next to the values that have been altered. We note that the new values do not differ by much from their original counterparts. The new balanced data set thus obtained is called Data Set 3. The matrix $\hat{\underset{\sim}{\Gamma}}$ now has the value

$$\hat{\underset{\sim}{\Gamma}} = \begin{bmatrix} .0300 & .0105 & -.0049 \\ .0105 & .0083 & -.0032 \\ -.0049 & -.0032 & .0025 \end{bmatrix} .$$

The corresponding value of M in (2.18) is .0677. This is a highly significant result since the lower .5% point of the statistic M from Beyer's (1968, p. 264) tables is $(.31509)^2 = .0993$. Thus, a minor change in Data Set 1 has led to a rejection of Model II. We can, therefore, clearly see the importance of checking the validity of Model II's assumptions concerning $\underset{\sim}{\Sigma}$ prior to using the standard techniques of analysis of variance for data analysis.

APPENDIX: PROOF OF THEOREM 2.1

(i) It is obvious that the $\underset{\sim}{w}_j$ in (2.10) are normally distributed ($j=1,2,\ldots,s$). Now, the two-way classification model in (1.1) can be written as

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_{n..} + \underset{\sim}{X}_1 \underset{\sim}{\alpha} + \underset{\sim}{X}_2 \underset{\sim}{\beta} + \underset{\sim}{X}_3 (\underset{\sim}{\alpha\beta}) + \underset{\sim}{\varepsilon}, \tag{A.1}$$

where $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, and $\underset{\sim}{X}_3$ are matrices of zeros and ones of orders $n.. \times r$, $n.. \times s$, and $n.. \times rs$, respectively; $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, and $(\underset{\sim}{\alpha\beta})$ are vectors whose elements are the α_i, β_j, and $(\alpha\beta)_{ij}$, respectively. The variance-covariance matrix of $\underset{\sim}{y}$ is

$$\underset{\sim}{\Delta} = [\underset{\sim}{X}_2 : \underset{\sim}{X}_3] \operatorname{Var}(\underset{\sim}{\psi}) [\underset{\sim}{X}_2 : \underset{\sim}{X}_3]' + \sigma_\varepsilon^2 \underset{\sim}{I}_{n..}, \tag{A.2}$$

where $\underset{\sim}{\psi} = [\underset{\sim}{\beta}', (\underset{\sim}{\alpha\beta})']'$. Let $\bar{\underset{\sim}{y}} = (\bar{\underset{\sim}{y}}_1', \bar{\underset{\sim}{y}}_2', \ldots, \bar{\underset{\sim}{y}}_s')'$ be the vector of cell sample means, where $\bar{\underset{\sim}{y}}_j$ ($j=1,2,\ldots,s$) is defined in (1.3). This vector can be expressed as

$$\bar{\underset{\sim}{y}} = \underset{\sim}{D}\, \underset{\sim}{y}, \tag{A.3}$$

where $\underset{\sim}{D}$ is the direct sum

$$\underset{\sim}{D} = \bigoplus_{i,j} (\underset{\sim}{1}_{n_{ij}}' / n_{ij}). \tag{A.4}$$

Let us now consider the matrix $\underset{\sim}{R}$ defined in (2.6). From (2.8) and (2.9) we can write

$$\underset{\sim}{R} = \sum_{j=1}^{s+1} \underset{\sim}{C}_j \underset{\sim}{C}_j' , \tag{A.5}$$

where

$$\begin{aligned} \underset{\sim}{C}_j' \underset{\sim}{C}_j &= \underset{\sim}{I}_r, \quad j=1,2,\ldots,s+1 \\ \underset{\sim}{C}_j' \underset{\sim}{C}_{j'} &= \underset{\sim}{0}, \quad j \neq j'. \end{aligned} \tag{A.6}$$

It is easy to verify that

$$\underset{\sim}{DR} = \underset{\sim}{0}, \; \underset{\sim}{RX}_2 = \underset{\sim}{0}, \; \underset{\sim}{RX}_3 = \underset{\sim}{0}. \tag{A.7}$$

From (A.2) and (A.7) we conclude that

$$\underset{\sim}{D} \, \underset{\sim}{A} \, \underset{\sim}{R} = \underset{\sim}{0}. \tag{A.8}$$

Also, from (A.5), (A.6), and (A.8) we have

$$\underset{\sim}{D} \, \underset{\sim}{A} \, \underset{\sim}{C}_j = 0, \quad j=1,2,\ldots,s. \tag{A.9}$$

Moreover,

$$\mathrm{Cov}(\bar{\underset{\sim}{y}}, \underset{\sim}{y}'\underset{\sim}{C}_j) = \underset{\sim}{D}\,\underset{\sim}{\Lambda}\,\underset{\sim}{C}_j, \quad j=1,2,\ldots,s. \tag{A.10}$$

Formulas (A.9) and (A.10) imply that $\bar{\underset{\sim}{y}}$ and $\underset{\sim}{C}_j' \underset{\sim}{y}$, and hence $\bar{\underset{\sim}{y}}_\ell$ and $\underset{\sim}{C}_j' \underset{\sim}{y}$, are statistically independent for ℓ, $j=1,2,\ldots,s$. Furthermore, under Model I, the $\bar{\underset{\sim}{y}}_j$ $(j=1,2,\ldots,s)$ are statistically independent. The $\underset{\sim}{C}_j' \underset{\sim}{y}$ $(j=1,2,\ldots,s)$ are also statistically independent due to the fact that

$$\mathrm{Cov}(\underset{\sim}{C}_j' \underset{\sim}{y}, \underset{\sim}{y}' \underset{\sim}{C}_{j'}) = \underset{\sim}{C}_j' \Lambda \underset{\sim}{C}_{j'} = 0, \quad j \neq j', \tag{A.11}$$

since

$$\underset{\sim}{C}_j' \underset{\sim}{X}_2 = \underset{\sim}{0}, \; \underset{\sim}{C}_j' \underset{\sim}{X}_3 = \underset{\sim}{0}, \; j=1,2,\ldots,s, \tag{A.12}$$

$$\underset{\sim}{C}_j' \underset{\sim}{C}_{j'} = \underset{\sim}{0}, \quad j \neq j',$$

by (A.5), (A.6), and (A.7). From the above arguments we conclude that the $\underset{\sim}{w}_j$ $(j=1,2,\ldots,s)$ in (2.10) are statistically independent.

(ii) From (A.1) we have

$$E(\underset{\sim}{w}_j) = E(\bar{\underset{\sim}{y}}_j)+(\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j)^{\frac{1}{2}} \underset{\sim}{C}_j'(\mu \underset{\sim}{1}_{n..} + \underset{\sim}{X}_1 \underset{\sim}{\alpha}), \quad j=1,2,\ldots,s. \tag{A.13}$$

Now, in addition to (A.7), the matrix $\underset{\sim}{R}$ satisfies

$$\underset{\sim}{R}\, \underset{\sim}{1}_{n..} = \underset{\sim}{0}, \quad \underset{\sim}{R}\underset{\sim}{X}_1 = \underset{\sim}{0}. \tag{A.14}$$

From (A.5), (A.6), and (A.14) we thus have

$$\underset{\sim}{C}_j' \underset{\sim}{1}_{n..} = \underset{\sim}{0}, \quad \underset{\sim}{C}_j' \underset{\sim}{X}_1 = \underset{\sim}{0}, \quad j=1,2,\ldots,s. \tag{A.15}$$

From (2.2), (A.13), and (A.15) we conclude that

$$E(\underset{\sim}{w}_j) = E(\bar{\underset{\sim}{y}}_j) = \mu \underset{\sim}{1}_r + \underset{\sim}{\alpha}, \quad j=1,2,\ldots,s.$$

(iii) In part (i) it was shown that $\bar{\underset{\sim}{y}}_j$ and $\underset{\sim}{C}_j' \underset{\sim}{y}$ are statistically independent for $j=1,2,\ldots,s$. It follows that

$$\mathrm{Var}(\underset{\sim}{w}_j) = \mathrm{Var}(\bar{\underset{\sim}{y}}_j) + (\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j)^{\frac{1}{2}} \underset{\sim}{C}_j' \underset{\sim}{\Lambda} \underset{\sim}{C}_j (\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j)^{\frac{1}{2}}, \quad j=1,2,\ldots,s. \tag{A.16}$$

But, from (A.6) and (A.12) we note that

$$\underset{\sim}{C}_j' \underset{\sim}{\Lambda} \underset{\sim}{C}_j = \sigma_\varepsilon^2 \underset{\sim}{I}_r, \quad j=1,2,\ldots,s. \tag{A.17}$$

Hence, for $j=1,2,\ldots,s$,

$$\begin{aligned}\mathrm{Var}(\underset{\sim}{w}_j) &= \mathrm{Var}(\bar{\underset{\sim}{y}}_j) + \sigma_\varepsilon^2(\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j)\\ &= \underset{\sim}{\Sigma} + \sigma_\varepsilon^2 \underset{\sim}{K}_j + \sigma_\varepsilon^2(\lambda \underset{\sim}{I}_r - \underset{\sim}{K}_j), \text{ by (2.3)},\\ &= \underset{\sim}{\Sigma} + \lambda\sigma_\varepsilon^2 \underset{\sim}{I}_r\\ &= \underset{\sim}{\Gamma}.\end{aligned}$$

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TABLE 1. THE LOGARITHMS OF THE PERMEABILITIES IN SECONDS

(Hald 1952, p. 472)

Day	Machine		
	1	2	3
1	1.404*	1.306	1.932*
	1.346	1.628	1.674
	1.618	1.410*	1.399(1.10)
2	1.447	1.241*	1.426(1.126)
	1.569	1.185	1.768
	1.820	1.516	1.859*
3	1.914	1.506	1.382
	1.477	1.575*	1.690
	1.894	1.649	1.361
4	1.887	1.673	1.721
	1.485	1.372*	1.528
	1.392	1.114	1.371
5	1.772*	1.227	1.320
	1.728	1.397*	1.489
	1.545(1.245)	1.531	1.336(1.691)
6	1.665	1.404	1.633
	1.539	1.452*	1.612
	1.680	1.627	1.359
7	1.918	1.229*(1.729)	1.328
	1.931	1.508	1.802
	2.129	1.436	1.385
8	1.845	1.583	1.689
	1.790	1.627	2.248(1.646)
	2.042	1.282	1.795(1.195)
9	1.540	1.636	1.703
	1.428	1.067	1.370
	1.704	1.384	1.839

Note: Values marked with an asterisk are deleted for Data Set 2. Values in parentheses replace the altered values to the left for Data Set 3.

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